Theorem 2: The inverse system (4) and (5) is controllable if the original system is controllable.

Before proceeding with the proof a few remarks are in order. The proof depends heavily on Vandevenne's [4] multiple delay, pointwise controllability criterion. Unfortunately, for the general case (i.e., arbitrary but finite number of delays) the criterion is somewhat complex notationally. In order to make the proof of Theorem 2 more transparent it will be assumed that (4) has two delays (i.e., $r = 2$). Organization of the proof when the number of delays is greater than two should be clear.

Proof of Theorem 2: If $r = 2$, then (4) has three coefficient matrices. Label them F_0, F_1 , and F_2 , respectively. This inverse system will be pointwise controllable for some $t > 0$ if [4]

$$\text{rank} [Q_{11}^1, Q_{11}^2 Q_{21}^2 Q_{12}^2, \dots, Q_{11}^n \dots Q_{1n}^n Q_{21}^n \dots Q_{2,n-1}^n \dots Q_{n1}^n] = n \quad (6)$$

where

$$\begin{aligned} Q_{11}^1 &= B \\ Q_{ij}^{k+1} &= F_0 Q_{ij}^k + F_1 Q_{i-1,j}^k + F_2 Q_{i,j-1}^k \\ Q_{ij}^k &= 0 \text{ for } k = 0, \quad i \text{ or } j = 0, i \text{ or } j > k. \end{aligned}$$

For the case of one delay (i.e., $r = 1$), (6) reduces to the well-known condition [2]

$$\text{rank} [Q_1^1, Q_1^2 Q_2^2, \dots, Q_1^n \dots Q_n^n] = n.$$

It is easy to show that $Q_i^k \equiv Q_{i1}^k$. Thus, (6) may be written as

$$\text{rank} [Q_1^1, Q_1^2 Q_2^2, \dots, Q_1^n \dots Q_n^n, Q_{12}^2, \dots, Q_{n-1,2}^n] = n. \quad (7)$$

Clearly, if the triple (A_0, A_1, B) is pointwise controllable (i.e., (1) is pointwise controllable) then (F_0, F_1, B) is pointwise controllable and the rank condition of (7) is satisfied. This completes the proof.

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An Extension of a Theorem by Rosenbrock

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Abstract—This note contains an extension of a theorem by Rosenbrock concerning the factorization of $\det(I + FG)$ when F is not necessarily diagonal.

The purpose of this note is to extend Theorem 1 of Rosenbrock [1] to the case where the feedback matrix F is *not necessarily diagonal*. The proof of this extension will be along the lines drawn by Barnett [2].

Using similar notation as in [1] and [2] let G , the *open-loop transfer matrix*, be an $n \times n$ matrix whose elements are rational functions, and let F , the *feedback matrix*, be an $n \times n$ matrix of real numbers. Furthermore, let, for $i = 1, \dots, n$,

$$F_i^T = [f_{i1}^T, f_{i2}^T, \dots, f_{in}^T, 0, \dots, 0] \quad (1)$$

$$K_i^T = [0, \dots, 0, f_{i1}^T, 0, \dots, 0] \quad (2)$$

$$H_i = G(I + F_{i-1}G)^{-1} \quad (3)$$

where f_i is the i th row of the $n \times n$ matrix F . Also let h_i be the i th column of H_i .

Theorem 1 of Rosenbrock [2] states that for F diagonal

$$\det(I + FG) = \prod_{i=1}^n (1 + f_{ii}h_{ii}) \quad (4)$$

where f_{ii} and h_{ii} are the (i,i) th element of F and H_i , respectively.

This result will be extended here to the case where F is *not necessarily diagonal*. Proceeding as in [2],

$$|I + K_i H_i| = 1 + f_i h_i \quad (5)$$

where, in the present note $f_i h_i$ is the scalar product of the vectors f_i^T and h_i . Continuing along the lines of [2], one can see that all steps of the proof are valid for F_i and K_i as defined in (1) and (2), and the assumption made in [1] and [2] that F is diagonal is not essential.

Thus the extension of Theorem 1 is

$$|I + FG| = \prod_{i=1}^n (1 + f_i h_i). \quad (6)$$

Note that an analogous result to (6), namely,

$$|I + GF| = \prod_{i=1}^n (1 + \tilde{h}_i \tilde{f}_i) \quad (7)$$

can also be obtained. Here $\tilde{f}_i$ is the i th column of F , and $\tilde{h}_i$ is the i th row of $\tilde{H}_i$, where

$$\tilde{H}_i = (I + G\tilde{F}_{i-1})^{-1} G \quad (8)$$

and

$$\tilde{F}_i = [\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_i, 0, \dots, 0]. \quad (9)$$

The formulas (6) and (7) in their present context, can be used to extend Mayne's design algorithm [3] to the case where the feedback matrix F is not necessarily diagonal.

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Eigenvalues of Partitioned Matrices "Generated" by a Matrix with Known Eigenvalues

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Abstract—A method is shown which allows to compute the eigenvalues of an mn -dimensional square matrix F from the eigenvalues of an n -dimensional matrix A , if the m^2 submatrices of F are describable by convergent power series of A . As an illustration, matrices arising in a multivariable feedback control system are considered.

I. INTRODUCTION AND STATEMENT OF THE THEOREM

Stability analysis in the design of linear time-invariant multivariable control systems may lead to the problem of determining the eigenvalues of an mn -dimensional matrix of the form

$$F_{mn}(A) = [f_{ij}(A)], \quad i, j = 1, 2, \dots, m \quad (1)$$

where the m^2 elements (submatrices) $f_{ij}(A)$ are functions which can be expressed as convergent power series of the n -dimensional matrix A .

It appears that the following theorem can be used with advantage:

The characteristic polynomial of (1) can be written in the form

$$\det(\lambda I_{mn} - F_{mn}(A)) = \prod_{k=1}^s \{\det(\lambda I_m - F_m(\lambda_k))\}^{\nu_k} \quad (2)$$

where $F_m(\lambda_k)$ is an m -dimensional matrix with the elements $f_{ij}(\lambda_k)$ and the λ_k are the s distinct eigenvalues of A with multiplicity ν_k .

II. PROOF

Since this theorem is not mentioned in textbooks (see for example [1], [2]), a short proof is as follows:

The matrix A can be transformed into a Jordan canonical (quasi-diagonal) form by means of

$$J = T^{-1} A T = \text{diag}\{J^{\nu_{n1}}(\lambda_1), \dots, J^{\nu_{11}}(\lambda_1), \dots, J^{\nu_{ik}}(\lambda_k), \dots, J^{\nu_{ns}}(\lambda_s), \dots, J^{\nu_{1s}}(\lambda_s)\} \quad (3)$$

where J^{ν}_{ik} is a ν_{ik} -dimensional Jordan matrix of λ_k ;

$$\sum_{i=1}^n \nu_{ik} = \nu_k = \text{multiplicity of } \lambda_k \text{ and } \sum_{k=1}^s \nu_k = n = \text{dimension of } A.$$

Matrix (1) can therefore be written

$$F_{mn}(A) = \text{diag} \{T^{-1}, T^{-1}, \dots, T^{-1}\} F_{mn}(J) \text{diag} \{T, T, \dots, T\}. \quad (4)$$

An mn -dimensional permutation matrix P transforms now $F_{mn}(J)$ into quasidiagonal form

$$F_{mn}^* = P^{-1}F_{mn}(J)P = \text{diag}\{\dots, F_{\nu_{ik}}(\lambda_k), \dots\} \quad (5)$$

where the $F_{\nu ik}(\lambda_k)$ are quasitriangular matrices of the form

$$F_{\nu ik}(\lambda_k) = \begin{bmatrix} F_m'(\lambda_k) & | & F_m''(\lambda_k) & | & \frac{1}{(\nu_{ik}-1)!} F_m^{(\nu_{ik-1})}(\lambda_k) \\ - & + & - & + & - \\ F_m(\lambda_k) & | & F_m'(\lambda_k) & | & \dots \\ - & + & - & + & - \\ & | & F_m(\lambda_k) & | & \dots \\ - & + & - & + & - \\ & | & & | & \dots \\ - & + & - & + & - \\ & | & & | & F_m(\lambda_k) \end{bmatrix}. \quad (8)$$

(the submatrices are $F_m(\lambda_k)$ and its derivatives with respect to λ_k). Since the characteristic polynomial is invariant with respect to linear transformation:

$$\begin{aligned} \det(\lambda I_{mn} - F_{mn}(A))' &= \det(\lambda I_{mn} - F_{mn}^*) \\ &= \prod_{\nu_{ik}} \{ \det(\lambda I_{m\nu_{ik}} - F_{\nu_{ik}}(\lambda_k)) \} \end{aligned}$$

and since (6) is quasitriangular:

$$\det(\lambda I_{mn} - F_{mn}(A)) = \prod_{k=1}^s \{\det(\lambda I_m - F_m(\lambda_k))\}^{\nu_k}.$$

Q.E.D.

III. ILLUSTRATIVE EXAMPLE

Consider the servosystem of Fig. 1. The plant is determined by the n -dimensional nonsingular matrices A , B , and C (see for example [3]). The eigenvalues of A are known. The integral controller is designed in such a way that for a ramp-function reference input $r = (t/T_R) \cdot r_0$ the output vector satisfies the following condition:

$$\lim_{t \rightarrow \infty} \mathbf{y}(t) = \frac{t - T_D}{T_R} \mathbf{r}_0, \quad (7)$$

i.e., for $t \rightarrow \infty$ all output variables have the same time delay with respect to the reference input $r(t)$. This condition leads to the integral controller

$$T_i = -T_D C A^{-1} B. \quad (8)$$

The eigenvalues of the closed-loop system are obtained by considering the equation

$$\dot{x} = Ax + BT_i^{-1} \int_0^t (r - Cx) dt. \quad (9)$$

After differentiation and substituting (8) in (9) one gets the following homogeneous equation:

$$\ddot{x} = A\dot{x} + \frac{1}{T_D}Ax \quad (10)$$

Fig. 1.

which can be written

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & I \\ A/T_D & A \end{bmatrix} \cdot \begin{bmatrix} x \\ \dot{x} \end{bmatrix}. \quad (11)$$

According to theorem (2) the eigenvalues of (11) are given by the roots of the s quadratic equations

$$\lambda^2 - \lambda\lambda_k - \lambda_k/T_D = 0, \quad k = 1, 2, \dots, s \quad (12)$$

where s is the number of distinct eigenvalues λ_k of A . These roots have the same multiplicity ν_k as the corresponding λ_k of A .

IV. CONCLUSIONS

Whenever an mn -dimensional matrix can be expressed in the form (1), its eigenvalues can be computed from the eigenvalues λ_k of an n -dimensional matrix by solving $s \leq n$ m th-order equations of the form

$$\det (\lambda I_m - F_m(\lambda_k)) = 0.$$

The factorization of the characteristic equation improves considerably the precision of numerical computation.

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Eigenvalue Assignment for Real Matrices of Specified Form

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Abstract—The purpose of this note is to report on various existence conditions found in connection with the problem of assigning the eigenvalues of a real n th-order matrix constrained by the specification of certain zero entries. This problem finds applications in many areas, but it was originally motivated by attempts to obtain desired responses from an n th-order dynamic system governed by a set of linear homogeneous differential equations of the form $\dot{x} = Ax$ where A is the constant real system matrix to be found.

This note addresses itself to the existence conditions that have been found governing the existence of a real matrix with specified zero entries representative of a specified complex conjugate set of eigenvalues. Origins and details of this problem are reported in greater detail in [1] and [2]. For the general case of a real matrix A representative of an actual physical system, it should be noted that A will usually be fairly sparse. For this reason we force these zero entries to remain zero, and after specifying the desired eigenvalues we seek the matrix satisfying these conditions.

NECESSARY CONDITIONS FOR THE GENERAL CASE

Theorem: Let $A = [a_{ij}]$ be an $n \times n$ real matrix with eigenvalues $\lambda_i = \alpha_i + j\beta_i$ which occur in conjugate pairs when the β_i are nonzero. Then

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